Midterm

MTH 616, Semester 2, 2021-22

Duration: 2 hours

Maximum Points: 50

Directions: Please ensure that you sufficiently explain and justify all intermediate arguments leading to any conclusions that you may draw along the way. Each statement (or argument) in your solution must be clearly explained, and must be devoid of any logical gaps or inconsistencies.

1. Let $f : X \to X$ be a continuous map, and let M_f denote its mapping cylinder. (For a map $f : X \to Y$, M_f is the quotient space $X \times I \sqcup Y/(x, 1) \sim f(x)$, for all $x \in X$)

   (a) For $n \geq 1$ and $m > 1$, let $f : S^n \to S^n$ be a degree m map. Show that M_f cannot retract to S^n.

   (b) Using (a), show that the Möbius band cannot retract onto its boundary circle. [5+5]

2. Compute the homology groups of the following spaces.

   (a) The quotient of S^2 obtained by identifying the north and south poles.

   (b) The quotient of S^2 under by identifications $x \sim -x$ for each x in the equator $S^1 \subset S^2$. [10+10]

3. Let A, B be a pair of subspaces of a space X such that $X = A^\circ \cup B^\circ$. Then there exists a long exact sequence of homology groups known as the *Mayer-Vietoris sequence* given by

   $$\ldots \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \ldots \to H_0(X) \to 0,$$

   which is associated with the short exact sequence of chain complexes

   $$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \to 0,$$

   where $\phi(x) = (x, -x)$, $\psi(x, y) = x + y$, and ∂ is induced by the usual boundary map $\partial : C_n(X) \to C_{n-1}(X)$. The Mayer-Vietoris sequence for homology can be viewed as an analog of the Seifert van Kampen Theorem for fundamental groups.

   (a) Using the Mayer-Vietoris sequence show that $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$ for all n. [Hint: SX is quotient space obtained by taking two copies of CX identified along X.]

   (b) Using the Mayer-Vietoris sequence, compute the singular homology groups for the Klien Bottle K. [Hint: K is the quotient space obtained by taking two copies of the Möbius band attached along their boundaries by the identity map.] [10+10]